Fractal Spectrum of a Quasi-Periodically Driven Spin System.

Italo Guarneri[1] and Maria Di Meo

Università di Milano, sede di Como, via Lucini 3,

22100 Como, Italy

Abstract

We numerically perform a spectral analysis of a quasi-periodically driven spin 1/2 system, the spectrum of which is Singular Continuous. We compute fractal dimensions of spectral measures and discuss their connections with the time behaviour of various dynamical quantities, such as the moments of the distribution of the wave packet. Our data suggest a close similarity between the information dimension of the spectrum and the exponent ruling the algebraic growth of the 'entropic width' of wavepackets.

I. INTRODUCTION

The increasingly frequent apparition of Singular Continuous (S.C.) spectra in various quantum mechanical situations has attracted attention to the dynamical implications of such spectra. For example, for an electron moving in an incommensurate or disordered structure, singular continuous spectra typically result in a sort of pseudo-diffusive dynamics, which has a direct bearing on transport properties. If the spectrum is a multifractal, some of these properties depend on the value of certain fractal dimensions, according to quantitative rules, the precise assessment of which is an important theoretical task .

SC spectra have also been found in some periodically or quasi-periodically driven model systems, which were introduced in order to investigate the possibility of chaotic behaviour in quantum dynamics. [2] [3]. In particular, quasi-periodically driven spin systems have

been studied, in view of their formal simplicity. Various spectral types have been identified , on varying levels of evidence, for different systems in this class [4]. From the mathematical viewpoint, some quasi-periodically driven spin systems formally belong to a class of abstract dynamical systems, which is well-known in ergodic theory, and for some of which SC spectra have been rigorously proven to occur [6]. In a recent paper [5] a renormalization group analysis has been implemented, strongly supporting SC spectra for a wider class of spin systems than encompassed by available exact results. The formal simplicity of this class of systems makes a numerical analysis of their dynamics particularly convenient, so that they appear very well suited for the study of the dynamical implications of fractal spectra.

In this paper we describe a numerical spectral analysis for a particular model in this class, aimed at computing certain fractal dimensions, and at connecting them to asymptotic aspects of the dynamics. We use a technical approach, based on a discrete-time variant of Floquet theory, which has not been implemented before in this context, and which offers a twofold advantage. In the first place, if combined with a suitable scheme of rational approximation for an incommensuration parameter, it allows for a reliable computation of the spectral measures, at a relatively low computational cost. Second, the Floquet formulation allows for a meaningful analysis of the growth in time of the spread of wave-packets, which is a very efficient empirical marker for continuous spectra, and usually a more convenient one than the hitherto used decay of correlations.

We then analyze a number of dynamical features, related to the growth of wave-packets. We find that the exponents of growth of the moments of the wave packet are not directly related to the Hausdorff dimension of the spectrum, (which is 1, as shown by a simple rigorous argument). We also find that the exponent of growth of the "entropic" spread of the wavepacket is close to the numerically computed information dimension of the spectral measure. Finally, we shortly present a general bound for the growth of the information contents of a finite string of observations with the length of the string, in terms of the information dimension of the spectrum.

II. THE MODEL

Our model is a periodically kicked spin $\frac{1}{2}$ system, with the kicking strength depending quasi-periodically on time. We consider a function $\hat{S}(\varphi)$ from $[0, 2\pi]$ into the unitary, unimodular 2×2 matrices, given by:

$$\hat{S}(\varphi) = e^{ik\chi(\varphi)\hat{\sigma}_x} e^{i\hat{\sigma}_z} \tag{2.1}$$

where χ is a periodic real-valued function, to be specified later, k is a parameter, and $\hat{\sigma}_x, \hat{\sigma}_z$ are Pauli matrices. In $[0, 2\pi]$ we consider the shift $\tau_\alpha : \varphi \longmapsto \varphi + 2\pi\alpha(mod 2\pi)$, with α a fixed parameter. We define the dynamics of our model by first arbitrarily fixing a phase φ_0, and then prescribing that spinors $\vec{\psi}(t) = \begin{vmatrix} \psi_1(t) \\ \psi_2(t) \end{vmatrix}$ given at (discrete) time t evolve at time $t+1$ into:

$$\vec{\psi}(t+1) = \hat{S}(\tau_\alpha^t \varphi_0)\vec{\psi}(t) \tag{2.2}$$

At fixed φ_0, α, k the evolution of a given initial spinor is thus obtained by applying a sequence of unitary matrices, which can be either a periodic or a quasi-periodic one, depending on the arithmetic nature of the number α :if the latter is a rational, $\alpha = p/q$ with p, q mutually prime integers, the sequence is periodic with period q, otherwise it is quasi-periodic. We have chosen the function χ as the periodicized characteristic function of an interval I in $[0, 2\pi]$: $\chi(\varphi) = 1$ if $\varphi(mod 2\pi) \in I$, $\chi(\varphi) = 0$ otherwise. As we shall explain below, such a choice makes the numerical analysis very efficient. The sequence of unitary matrices defining the evolution is now uniquely defined by the symbolic trajectory of the chosen φ_0 associated with the shift τ_α and with the partition of $[0, 2\pi]$ defined by I and its complement. Since this partition consists of two sets, the symbolic sequence can be written as a binary sequence. In particular, if the length of the interval I is taken $2\pi\alpha$ and α , and α is the (inverse) Golden Ratio $(\sqrt(5) - 1)/2$, one obtains the Fibonacci sequence.

III. SPECTRAL MEASURES.

There are two possible approaches to the spectral analysis of the system (2.1). The first consists in considering (2.1) as a classical dynamical system, the state of which is defined by a couple $\phi, \vec{\psi}$ where ϕ is an angle and $\vec{\psi}$ is a normalized spinor; the one-step evolution of the system is given by:

$$(\varphi, \vec{\psi}) \longmapsto (\tau_\alpha \varphi, \hat{S}(\varphi)\vec{\psi}) \tag{3.1}$$

The system thus defined belongs to a class of dynamical systems known as "skew-products" [6]. Spectral measures are defined from the Fourier transform of correlation functions; in particular, given an initial spinor $\vec{\psi}(0)$, the corresponding spectral measure $d\mu$ is defined by:

$$R(t) = \int_0^{2\pi} e^{it\lambda} d\mu(\lambda) \tag{3.2}$$

where $R(t)$ is the correlation function:

$$R(t) = \lim_{T\to\infty} \frac{1}{T} \sum_{s=0}^{T} \left\langle \vec{\psi}(s), \vec{\psi}(s+t) \right\rangle \tag{3.3}$$

where $\langle , \rangle$ is the scalar product in C^2. Spectral measures defined in this way will be termed "dynamical" measures in the following.

Another definition of spectral measure rests on a generalization of Floquet theory, which is obtained on imbedding the nonautonomous quantum dynamics 2.2 in an autonomous dynamics, defined in a larger Hilbert space. This is done as follows. We consider the phase φ as a new dynamical variable, and thereby consider state vectors $\Psi = \vec{\psi}(\varphi)$ in the Hilbert space $\mathcal{H} = L^2([0, 2\pi]) \otimes C^2$. In this space we consider the discrete unitary group generated by the unitary Floquet operator:

$$(\mathcal{S}\vec{\psi})(\varphi) = \hat{S}(\varphi)\vec{\psi}(\tau_\alpha^{-1}\varphi) \tag{3.4}$$

If $\vec{\psi}(\varphi)$ is regarded as a curve in the phase space of the dynamical system (3.1), then (3.4) specifies the evolution of this curve under the dynamics (3.1). Our second definition

of a spectral measure is just the usual one for unitary operators, applied to the operator (3.4). Such spectral measures will be termed "Floquet measures". The connection between dynamical and Floquet measures will be discussed separately for the incommensurate and the commensurate cases. For the time being, we shall point out a simple property of the spectrum of $\mathcal{S}$, which is valid in both cases. On defining in $\mathcal{H}$ the unitary operator $\mathcal{U}\vec{\psi}(\varphi) = e^{i\varphi}\vec{\psi}(\varphi)$ we find:

$$\mathcal{U}^\dagger \mathcal{S} \mathcal{U} = e^{-2\pi i\alpha}\mathcal{S}$$

which implies that the spectrum of $\mathcal{S}$ is invariant under the shift τ_α : that is, if a point of the unit circle belongs in the spectrum of $\mathcal{S}$, so does the whole orbit of that point under τ_α .

It is worth remarking that the operator $\mathcal{S}$ can be interpreted as the Floquet operator for a linear kicked rotator endowed with spin; the occurrence of SC spectra for a spinless linear kicked rotator has been discussed in [2].

IV. THE COMMENSURATE CASE.

In the commensurate case $\alpha = p/q$, the model is a periodically driven one, and its dynamics can be understood, as usual, from a spectral analysis of the one-period evolution operator. The state vector at times multiple of the period q is found from:

$$\vec{\psi}(nq) = \hat{T}^n(\varphi_0)\vec{\psi}(0)$$

where

$$\hat{T}(\varphi_0) = \prod_{t=0}^{q-1} \hat{S}(\tau_\alpha^t \varphi_0) \tag{4.1}$$

(the product being ordered from right to left) yields the evolution over one period. This operator does not depend on time; thus the dynamics, read at integer multiples of the period, is given by a discrete unitary group, generated by a fixed unitary (4.1). Being a finite matrix, the latter has obviously a pure discrete spectrum, consisting of two conjugate eigenvalues $z_1 = e^{i\lambda}$, $z_2 = z_1^* = e^{-i\lambda}$. The dynamics is therefore recurrent.

From (3.4)(4.1) we find:

$$(\mathcal{S}^q\vec{\psi})(\varphi) = \hat{T}(\varphi)\vec{\psi}(\varphi) \tag{4.2}$$

that is, $\mathcal{S}^q$ is a fibered operator. Its fibers are precisely the 2×2 matrices$\hat{T}(\varphi)$, and its spectrum is the range of the functions $z_1(\varphi), z_2(\varphi)$ giving the eigenvalues of $\hat{T}(\varphi)$. It turns out that, when the size of the interval I is not a multiple of $2\pi/q$, both functions have exactly two values in their range: otherwise, they have just one. This fact allows for a decisive simplification of the numerical analysis, and is proved in Appendix A; it stems from the finiteness of the range of the function χ.

Therefore the spectrum of $\mathcal{S}^q$ consists of a finite number $2M$ of eigenvalues $z_N = e^{i\Lambda_N}, (N = 1, ...2M)$, in complex conjugate pairs, with $M = 2$ or $M = 1$. The spectrum of $\mathcal{S}$ is then a subset of the set of the $2Mq$ complex q-th roots of the z_N. The corresponding eigenphases are:

$$\lambda_{j,N} = \frac{\Lambda_N}{q} + \frac{2\pi j}{q} \tag{4.3}$$

The spectrum of $\mathcal{S}$ is discrete, i.e., the Floquet dynamics in the extended Hilbert space has the same spectral character as the proper dynamics. It should be noted that the fact that the dynamical measure and the Floquet one are of the same type is, in the commensurate case, a nongeneric feature , connected with the choice of a finitely valued function χ in (2.2). On choosing a nonconstant analytic function χ, an absolutely continuous spectrum of $\mathcal{S}$ would be found instead, in contrast to the character of the dynamics, that would be still recurrent.

V. THE INCOMMENSURATE CASE.

The spectral measure of a vector $\Psi =\vec{\psi}(\varphi)$ with respect to the operator (3.4) is defined as the Fourier transform of the correlation:

$$C(t) = \left\langle \Psi, \mathcal{S}^t\Psi \right\rangle_H = \int d\varphi \left\langle \vec{\psi}(\varphi), S(\varphi)S(\tau_\alpha^{-1}\varphi)...S(\tau_\alpha^{-t}\varphi)\vec{\psi}(\tau_\alpha^{-1-t}\varphi) \right\rangle \tag{5.1}$$

In the incommensurate case, the well-known ergodic property holds:

$$\frac{1}{2\pi}\int d\varphi g(\varphi) = \lim_{T\to\infty}\frac{1}{T}\sum_{s=0}^{T} g(\tau_\alpha^{-s}\varphi_0) \tag{5.2}$$

for any summable function g,and for almost every φ_0. Then the integral over φ in (5.1) can be computed as follows:

$$C(s) = \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T}\left\langle \vec{\psi}(\tau_\alpha^{-t}\varphi_0), \hat{S}(\tau_\alpha^{-t}\varphi_0)\hat{S}(\tau_\alpha^{-t-1}\varphi_0)...\hat{S}(\tau_\alpha^{-t-s}\varphi_0)\vec{\psi}(\tau_\alpha^{-1-t-s}\varphi_0)\right\rangle \tag{5.3}$$

Now we apply to both factors in the scalar product the unitary operator $\hat{S}(\varphi_0)\hat{S}(\tau_\alpha^{-1}\varphi_0)...\hat{S}(\tau_\alpha^{-t+1}\varphi_0)$ and take $\vec{\psi} = \vec{\psi}(0)$. Then, on comparing the result with (3.3) we find that

$C(t) = R(t)$ except possibly for a set of zero measure of values of φ_0. Therefore, in the incommensurate case the dynamical spectral measure (3.3) coincides with the Floquet spectral measure.

In [5] a quite similar quasi-periodically kicked spin system was studied, which was reduced to a classical dynamical system for which singular continuity of the spectrum is a known mathematical result. It looks likely, on account of the closeness of our model to that one, that this result holds in our case, too; our numerical results, presented below, yield strong evidence in this sense.

Since in the incommensurate case the orbit of any point of the unit circle under the shift is dense in the circle, from the invariance of the spectrum under τ_α it follows that the spectrum (which is by definition a closed set) must coincide with the whole unit circle.

VI. NUMERICAL RESULTS

Our main focus is on the incommensurate case, with α given by the Golden Ratio. In most of our computations we have taken I an interval of length $2\pi/3$; in a few cases we have taken $2\pi\alpha$ instead. Unless explicitly stated, we will make reference to the first choice. We have obtained two types of numerical data: dynamical ones, obtained from directly simulating the Floquet dynamics, and spectral ones. We discuss the latter first.

A direct computation of the spectral measure from the Fourier transform of the numerically computed correlation function (3.3) involves certain subtleties, well-known in the spectral analysis of time series, connected with the necessity of appropriately weighting the tails of correlation functions [10]. A more convenient strategy for the present model is based on approximations of the Golden Ratio constructed via continued fraction expansion, which produces a well-known sequence of rational approximants. For such approximants we carefully compute the spectral measures, as we explain in Appendix B, and thus we obtain approximations of the true spectral measure, by means of the pure point measure (9.2).

A similar strategy has been widely used in the numerical investigation of other incommensurate spectral problems, such as e.g. the Harper and the Kicked-Harper model [3]. It is worth remarking, however, that in those cases, unlike the present one, the commensurate approximations have an *absolutely continuous* band spectrum, that is usually taken as a covering of the limit spectrum.

For the obtained measures we have performed a fractal analysis, aimed at the determination of the fractal dimensions D_q; the Hausdorff dimension D_H is 1 in the incommensurate case, because the spectrum in that case is the whole unit circle. In our computations we have used the definitions,

$$D_1 = \lim_{\delta \to 0} \frac{\sum_i \mu_i \ln \mu_i}{\ln \delta}; \qquad D_q = \lim_{\delta \to 0} \frac{\ln \sum_i \mu_i^q}{(q-1) \ln \delta} \tag{6.1}$$

where the interval $[0, 2\pi]$ was partitioned into small intervals of size δ, the $i-$th of which receives a weight μ_i from the spectral measure.

D_q were obtained by a linear fit of the numerators in (6.1) vs the denominators in a suitable range of δ. A typical result is shown in fig.1. In no case could the value D_0 obtained from the spectrum of generalized dimensions be distinguished from $D_H = 1$. The correlation dimension D_2 is known to rule the decay of integrated correlations [9]:

$$\frac{1}{T} \sum_{t=0}^{T-1} |C(t)|^2 \sim const. \cdot T^{-D_2} \tag{6.2}$$

asymptotically for $T \to \infty$. We have used this exact result to adjust our numerical computation of the other dimensions; in fact, since the approximating measure is a pure-point

one, the dimensions of the limit measure must be read off a suitable range of not too small δ. Our appreciation of a reliable range for δ was based on the comparison of the rhs of the 2nd eq.(6.1) with the exponent γ of algebraic decay of correlation obtained from a direct computation of the latter (Fig. 2).

We have also computed a number of dynamical data, by a direct numerical simulation of the dynamics. In particular, we have analyzed the growth in time of the momenta I_α of the distribution of the Floquet wave-function on the Fourier basis. I_α are defined by:

$$\begin{array}{c} I_\alpha(t) = \sum_{n=-\infty}^{+\infty} |n|^\alpha \, p_n(t) \\ p_n(t) = \frac{1}{2\pi t} \sum\limits_{s=0}^{t-1} \sum\limits_{j=1}^{2} \left| \int\limits_0^{2\pi} d\varphi e^{in\varphi} \psi_j(\varphi, s) \right|^2 \end{array} \tag{6.3}$$

Notice that moments of order $\alpha \geq 1$ diverge for $t > 0$, because the Fourier coefficients of the wave function behave as n^{-1} at large n, due to the discontinuity of χ .

Finally we have studied the growth in time of the average entropy $S(t)$:

$$S(t) = - \sum_{n=-\infty}^{+\infty} p_n(t) \ln p_n(t)$$

Examples of the dependence on time of the moments and of the entropy are given in figs.3,4. The moments increase according to a power law: $I_\alpha(t) \sim const.t^{\alpha\beta(\alpha)}$, and so does the "entropic number of states": $N(t) = \exp(S(t) \sim const.t^\sigma$. In Table 1 we summarize the values of the exponents γ,β, σ and the values of the D_1, D_2 for several choices of the kicking strength k.

The specified errors are those involved in linear fits of bilogarithmic plots used to compute fractal dimensions or growth exponents; they do not include numerical errors in computing the data themselves, therefore they to some extent underestimate the real errors . Generally speaking, it is usually difficult to get precise estimates for the dynamical growth exponents associated with fractal spectra, partly because of finite-basis effects coming into play at large times, and much more because the curves of growth display a characteristic pattern of kinks . In cases in which such seemingly log-periodic structures were particularly evident, the growth exponent was obtained directly by drawing a straight line through a sequence of

maxima. In such cases the fitting error, not specified in Table 1, is on the order of the first missing digit.

VII. DISCUSSION.

Some general exact results are known about the long-time properties of the quantum dynamics in the presence of a SC spectrum. First of all, integrated correlations like (6.2) must tend to zero, and momenta I_α must diverge in the limit $t \to \infty$ in all cases when the spectrum is purely continuous; furthermore, if the spectral measure is a fractal one, with a correlation dimension D_2, and information dimension D_1, then (6.2) holds for the decay of correlations [9], and $\beta(\alpha) \geq D_1$ [7]. Data in Table 1 are fully consistent with these exact estimates (as mentioned above, the estimate (6.2) was actually used to adjust our numerical method). In addition, more or less heuristic arguments relating more precisely the exponents $\beta(\alpha)$ to multifractality have been attempted. One such argument [8] has led to hypothesize that $\beta(\alpha) = D_H$, the Hausdorff dimension of the spectrum. Although consistent with numerical results from a number of models, this hypothesis has been called in question [11], and also recent numerical investigations have provided evidence that $\beta(\alpha)$ is not in general a constant but covers a continuous range of scaling exponents (multi-scaling). Our present data are fully consistent with the latter picture, because the observed values of $\beta(\alpha)$ differ from $D_H = 1$ (which was theoretically established and numerically confirmed) significantly more than the estimated numerical error. Somewhat smaller, but still significant, are the differences observed in the values of $\beta(\alpha)$ obtained for different values of α.

Two interesting facts emerging from table 1 are (i) the closeness of the values obtained for different values of k, which seems to indicate that the fractal structure of the spectrum is essentially determined by the quasi-periodic structure of the symbolic strings alone, and (ii) the closeness of the value of the "entropic" exponent σ to that of D_1. Although the difference of the two values was comparable to the relatively large numerical error, the actual agreement may be much better, because many of the data used in estimating D_1

appear not to have fully converged to the proper asymptotic regime. These data are in fact compared to the dynamical entropic exponent in Fig.5 ; the agreement "by eye" is there better than implied by Table 1. Whether this fact reflects a real connection between the two quantities is an interesting theoretical question, for which we have no answer for the time being.

Finally we shall mention a dynamical property, that, although not directly related to the above numerical results, has been recently introduced in essentially the same context as discussed here, with the aim of analyzing the possibly chaotic property of the evolution of the spin system. For a classical dynamical system in discrete time, with a compact phase space Ω, the "informational complexity" is introduced by considering a finite partition Π consisting of m subsets, and the associated symbolic dynamics. For any given integer time t, all the possible symbolic strings of length t define a partition Π^t of Ω^t. If a specific orbit of the system is chosen, then a probability can be attached to every class of this partition, defined as the frequency with which the given finite string appears in the infinite symbolic string of the given trajectory. The Shannon entropy $H(t)$ can then be defined, as well as the corresponding number of histories $N(t) = \exp(H(t))$. A chaotic behaviour is associated with an exponential growth of $N(t)$ in time.

In a recent paper [12] such an analysis has been numerically implemented for the case of a quasi- periodically driven spin system. In essence, Ω was taken as the compact variety of normalized spinors; a certain partition of Ω was constructed, and the growth of $H(t)$ with time was analyzed. A seemingly sub-exponential growth of $N(t)$ was observed, of the type $\exp(ct^\gamma)$ with $\gamma < 1$.

We shall here sketch a general argument, which establishes an upper bound for $N(t)$ in the presence of a fractal spectrum. At time t, Π^t is a partition of a $t.\nu$-dimensional space, where ν is the dimension of Ω; therefore $H(t) \leq t\nu \log m$. Now, all the strings $\psi(s+1), ...\psi(s+t)$ of length t which are observed in the evolution do indeed span a vector space of dimension t, because of the pure continuity of the spectrum; however, their "effective dimension" can be much less. It can in fact be proven [13] that the minimum dimension $d_\epsilon(t)$ of a subspace

which contains all the strings of length t within a maximum error ϵ asymptotically grows like t^{D_1}. This means that, apart from a small error of order ϵ, the entropy $H(t)$ is the same as the one computed over $\sim m^{c\nu t^{D_1}}$ classes. This leads to the upper bound $H(t) \leq const.t^{D_1}$.

Generally speaking, a direct numerical analysis of $N(t)$ seems difficult, because computing the frequency of strings long enough to reproduce the correct asymptotic regime requires an enormous computation time. In any case, for the special case of quasi-periodically driven spin systems, the above upper bound is but a very crude one: in fact a simple argument [14] indicates that the asymptotic growth of $N(t)$ cannot be faster than algebraic.

VIII. APPENDIX A

The matrix $\hat{T}(\varphi)$ is uniquely associated with a binary string of q digits, which specifies the symbolic periodic trajectory of φ. Suppose that the strings corresponding to two matrices $\hat{T}(\varphi), \hat{T}(\varphi')$ differ from each other merely by a cyclic permutation of digits. Then the two matrices differ from each other by a cyclic permutation of the $\hat{S}$ operators which enter as factors in the definition of the matrices themselves. The two matrices are then unitarily equivalent, and have therefore the same eigenvalues. We have thus reached the conclusion that the number of distinct values in the range of $z_1(\varphi)$ (and of $z_2(\varphi)$, as well) is equal to the number of nonequivalent symbolic strings of q digits, two strings being equivalent if they can be obtained from each other by a cyclic permutation. Thus in order to find this number we have to find the total number of symbolic strings of length q, and to divide it by q, which is the number of strings in an equivalence class. Let us denote by C_0 and C_1 (with $C_1 = I$) the two classes of the partition. Then the points φ which produce a given symbolic string $i_1, i_2, ...i_q$ are those belonging to the set $A_{i_1 i_2 ... i_q} = \tau_\alpha^{-1}(C_{i_1}) \cap ... \cap \tau_\alpha^{-q}(C_{i_q})$. The sets $A_{i_1 i_2 ... i_q}$ define a partition of $[0, 2\pi]$, and there are as many distinct symbolic strings of length q as are nonempty classes in this partition. It is easily seen that there are either $2q$ or q nonempty such classes, the latter case occurring if, and only if, I is a multiple of $2\pi/q$; in fact, on repeatedly applying the shift τ_α to the interval I, we get q distinct intervals,

whose endpoints make a set of $2q$ or q distinct points, depending on whether the length of I is a multiple of $2\pi/q$,or not. The sets $A_{i_1 i_2 \dots i_q}$ are precisely the disjoint intervals in which $[0, 2\pi]$ is divided by these $2q$ (resp., q) points, and their number is therefore $2q$ (resp., q). The number of nonequivalent strings is thus exactly 2 (resp., 1).

IX. APPENDIX B

In a commensurate case with $\alpha = p/q$ we get, from the definition (5.1) ::

$$C(t, \Psi) = \sum_{N=1}^{2M} e^{it\Lambda_N q^{-1}} \sum_{j=0}^{q-1} \left\| \hat{P}_{j,N} \Psi \right\|_H^2 e^{2\pi i t j q^{-1}} \tag{9.1}$$

where $\hat{P}_{j,N}$ is projection over the eigenspace of $\hat{c}$ corresponding to the eigenphase $\lambda_{j,N}$. The spectral measure in $[0, 2\pi]$ is pure point:

$$\sum_{N=1}^{2M} \sum_{j=0}^{q-1} \left\| \hat{P}_{j,N} \vec{\psi} \right\|_H^2 \delta(\lambda - \lambda_{j,N}) \tag{9.2}$$

In order to compute this measure we have to find the eigenphases $\lambda_{j,N}$ and the associated weights $p_{j,N} = \left\| \hat{P}_{j,N} \Psi \right\|_H^2$. The eigenphases are most easily computed, by diagonalizing the 2×2 matrix (4.1), for two different values of φ_0 (e.g., the two extremes of the interval I), and then using (4.3)(we assume that q is not a multiple of 3, for in that case just one value of φ_0 would be sufficient.). Concerning the weights, denoting $\hat{P}_N = \sum_j \hat{P}_{j,N}$, we have from (9.1):

$$e^{-it\Lambda_N q^{-1}} C(t, \hat{P}_N \Psi) = \sum_{j=0}^{q-1} p_{j,N} e^{2\pi i t j q^{-1}} \tag{9.3}$$

which is an exactly periodic function of the discrete time t with period q, so that the weights are quite easily and reliably computable via finite Fourier transform as soon as the lhs of (9.3) is known. Since Λ_N is computed as described above, we are left with the computation of the correlation of $\hat{P}_N \Psi$ at times $t = 0, ... q - 1$. To this end we first find $\hat{P}_N \Psi$ as follows: having diagonalized the matrix (4.1) for all values of φ_0 in a suitably thick grid, we let

$$\hat{P}_N \vec{\psi}(\varphi_0) = \left\langle \vec{u}_N(\varphi_0), \vec{\psi}(\varphi_0) \right\rangle \vec{u}_N(\varphi_0)$$

where $\vec{u}_N(\varphi_0)$ is defined as follows: it is the eigenvector of (4.1) with eigenvalue Λ_N, if the latter is an eigenvalue of (4.1) at the given φ_0, and it is the zero vector otherwise. The correlation of $\hat{P}_N\Psi$ is then found directly from the definitions (3.4),(5.1). The only approximation involved in this computation is the discretization of the scalar product in $\mathcal{H}$, which involves an integral over φ and is instead computed as a finite sum over the chosen finite grid used to discretize $[0, 2\pi]$.

We thank R.Artuso for valuable advise in computing fractal dimensions, and A.Vulpiani for an useful discussion.

REFERENCES

[1] also at I.N.F.N, sezione di Pavia, 27100 Pavia, Italy.

[2] J.Bellissard, in *Trends and Developments in the Eighties*, edited by S.Albeverio and P.Blanchard, World Scientific, Singapore 1985.

[3] C.Tang and M.Kohmoto, Phys.Rev.B**34**,2041 (1986); R.Artuso, G.Casati and D.L.Shepelyansky, Phys.Rev.Lett. **68**, 3826 (1992).

[4] Y.Pomeau, B.Dorizzi and B.Grammaticos, Phys.Rev.Lett. **56**, 681 (1986); B.Sutherland, *ibidem* **57**,770 (1986); R.Badii and P.F.Meier, *ibidem* **58**,1045 (1987); J.M.Luck, H.Orland, and U.Smilansky, J.Stat.Phys. **53**, 551 (1988); R.Graham, Europhys. Lett. **8**, 717 (1989; T.Geisel, Phys.Rev. A **41**, 2989 (1990).

[5] U.Feudel, A.S.Pikovsky, and M.A.Zaks, preprint Univ. Potsdam, July 1994.

[6] M.Queffelec, *Substitution Dynamical Systems - Spectral Analysis*, Springer Lect. Notes in Mathematics 1294, Berlin 1987, and references therein.

[7] I.Guarneri, Europhys. Lett. **10**, 95 (1989); *ibidem*, **21**, 729; I.Guarneri and G.Mantica, Ann. Inst. H.Poincare', in press (1993); I.Guarneri and G.Mantica, submitted to Phys.Rev.Lett. (1994); H.Hiramoto and M.Kohmoto, Int.J.Mod.Phys. B, **6**, 281 (1992).

[8] T.Geisel, R.Ketzmerick and G.Petschel, Phys.Rev.Lett. **66**, 1651 (1991).

[9] R.Ketzmerick, G.Petschel and T.Geisel, *ibidem* **69**, 695 (1992); M.Holschneider, Comm.Math.Phys. **160**, 457 (1994).

[10] R.Artuso, D.Belluzzo, G.Casati, Europhys. Lett. **25**, 181 (1994).

[11] M.Wilkinson and E.J.Austin, preprint (1993).

[12] A.Crisanti, M.Falcioni, G.Paladin, M.Serva and A.Vulpiani, Phys. Rev. E (1994) in press.

[13] I.Guarneri, unpublished.

[14] A.S.Pikovsky, private communication.

X. FIGURE CAPTIONS

Fig.1 - Spectrum of generalized fractal dimensions D_q, for $k = 10.5$. The rational approximant of the Golden Ratio used in this computation was 1597/2584.

Fig.2 - A bilogarithmic plot of $\sum \mu_i^2$ versus δ^{-1} (eqn.(6.1The slope of the straight line is the exponent of the correlation decay, eqn.(6.2). Here $k = 10.5$.

Fig.3 - Illustrating the growth of the moment $I_{1/2}$, for $k = 4$.

Fig.4 - Illustrating the growth in time of the entropy for $k = 10.5$. The slope of the straight line is the computed information dimension D_1.

Fig.5 - A bilogarithmic plot of $-\sum \mu_i \log \mu_i$ versus δ^{-1} (eqn.(6.1), for the case $k = 1$ of Table I. The slope of the dashed line is the entropic exponent σ, but the position of the line has been chosen arbitrarily.

TABLES

k	γ	D_2	σ	D_1	$\beta(1/2)$	$\beta(3/4)$
1	0.47	0.49±0.01	0.736±0.001	0.7±0.11	0.94	0.91
9	0.53	0.544±0.005	0.755±0.001	0.7±0.05	0.96	0.9
16	0.54	0.549±0.004	0.76±0.002	0.75±0.05	0.94	0.89
110.2	0.564	0.565±0.004	0.76±0.002	0.76±0.03	0.94	0.9

TABLE I. Some Fractal Dimensions and Dynamical exponents . The ratio of the size of the interval I to 2π was the Golden Ratio for the 1st row, 2/3 for the others.